

**ON MULTI-DIMENSIONAL RANDOM WALK MODELS  
APPROXIMATING SYMMETRIC SPACE-FRACTIONAL  
DIFFUSION PROCESSES**

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*Dedicated to the 60-th birthday of Professor Shavkat Alimov*

**Abstract**

In this paper the multi-dimensional analog of the Gillis-Weiss random walk model is studied. The convergence of this random walk to a fractional diffusion process governed by a symmetric operator defined as a hypersingular integral or the inverse of the Riesz potential in the sense of distributions is proved.

*Mathematics Subject Classification:* 26A33, 47B06, 47G30, 60G50, 60G52, 60G60

*Key Words and Phrases:* multi-dimensional random walk, Cauchy problem, fractional diffusion equation, pseudo-differential operators, fundamental solution, hypersingular integral

**1. Introduction**

In this paper we will study multidimensional random walk models approximating the Cauchy problem for fractional diffusion equations with symmetric spatial operator of fractional order. We will follow the method

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<sup>1</sup>Supported by German Academic Exchange Service (DAAD)

profitably used by Gorenflo and Mainardi for the study of scaling limits of discrete random walks, which was implemented in the one-dimensional case in the series of their works (see [6], [7] and references therein). They constructed a number of discrete random walk models approximating fractional diffusion processes with symmetric and non-symmetric Lévy-Feller fractional differential operators in the governing equation. The scaling weak limits of these models represent stable Lévy processes for the full range of scaling order  $\alpha \in (0, 2]$ .

Our treatment can be considered as a multi-dimensional generalization of the one-dimensional Gillis-Weiss model introduced first in [4] and studied in the recent paper [6]. The main purpose of this paper is to construct a multidimensional random walk models by choosing suitable transition probabilities, which approximate stable Lévy motions. Note that the extension of results from the one-dimensional case to the multi-dimensional case is not trivial and requires to use methods distinct from those, which were used in the one-dimensional case. In particular, the method used in [6], [7] involves summing of a formal Laurent series of a complex variable  $z$  and is no longer feasible in the multi-dimensional case. Instead we essentially use the symbolic calculus and properties of cubature formulas.

In the recent book [14] by M. Meerschaert and Scheffler the wide range of problems in multi-dimensional stochastic processes is highlighted, with emphasis on operator stable probability distributions, and there are also given comments on the historical development and indications of their different applications. Note that the necessary and sufficient conditions for a random vector to belong to the domain of attraction of nondegenerate nonnormal stable laws were found in [17] (see also [14]). Multi-dimensional random walk is often used in modeling various processes in different areas [12], [14]. See, for instance, [1] for applications to economics and [11] to finance, [2] for modeling of river flows and [19] for modeling of copepod behaviour of animals in zoology.

Our paper is organized as follows. In Section 2 we give auxiliary materials and introduce terminology that will be used in the paper. In Section 3 we mainly recall some properties of pseudo-differential operators considered in our previous paper [8] referring to it for details and lay out some elementary properties of symbols. These properties will be essentially used later in the study of the diffusion limits of random walks. In Section 4 we formulate the problem we are going to study in terms of random walk. In Section 5 we formulate the main result obtained in this paper.

## 2. Preliminaries

Let  $\mathbf{R}^N$  and  $\mathbf{Z}^N$  be the  $N$ -dimensional Euclidean space with coordinates  $\mathbf{x} = (x_1, \dots, x_N)$  and the  $N$ -dimensional integer-valued lattice with nodes  $j = (j_1, \dots, j_N)$ , respectively. Denote by  $x_j = (hj_1, \dots, hj_N)$ ,  $j \in \mathbf{Z}^N$ , the nodes of the uniform lattice  $\mathbf{Z}_h^N$  defined as  $(h\mathbf{Z})^N$  with a positive number  $h$ , the mesh width.

Suppose that a particle is located at the origin  $x_0 = \mathbf{0} = (0, \dots, 0)$  at the initial time  $t = 0$  and at time instances  $t_1 = \tau, t_2 = 2\tau, \dots, t_n = n\tau, \dots$  jumps moving through nodes of the lattice  $\mathbf{Z}_h^N$ . Let  $p_j$  be a probability of jumping from a point  $x_k \in \mathbf{Z}_h^N$  to a point  $x_{j+k} \in \mathbf{Z}_h^N$ , where  $j$  and  $k$  are in  $\mathbf{Z}^N$ . By this we automatically assume that the particle jumps are isotropic for all directions. The numbers  $p_j, j \in \mathbf{Z}^N$ , are called transition probabilities. They satisfy the following conditions of non-negativity and normalization:

$$(a) p_j \geq 0, j \in \mathbf{Z}^N; \quad (b) \sum_{j \in \mathbf{Z}^N} p_j = 1.$$

We denote by  $p_j$  the value of a discrete function  $p : \mathbf{Z}^N \rightarrow \mathbf{R}^1$  at a point  $j \in \mathbf{Z}^N$  (transition probabilities are also form a discrete function). For given two discrete functions,  $p$  and  $q$  we define a convolution  $p * q$  by the rule

$$(p * q)_j = \sum_{k \in \mathbf{Z}^N} p_k q_{j-k}, j \in \mathbf{Z}^N.$$

Also for a given discrete function  $p$  we define its Fourier transform (Fourier series)  $\hat{p}(\xi) = F[p](\xi), \xi \in \mathbf{R}^N$ , by the formula

$$\hat{p}(\xi) = \sum_{k \in \mathbf{Z}^N} p_k e^{-ik\xi},$$

and we call  $\hat{p}(-\xi)$  the characteristic function for  $p$ .

We will consider the solution  $u(t, x)$  of a fractional diffusion equation (see section 4) as a probability density (with respect to  $x$ ), namely for given time  $t > 0$  as the probability of sojourn of a diffusing particle at  $x \in \mathbf{R}^N$ . For the discrete random walk introduced above we use the notation  $y_j(t_n)$  for the (discrete) probability of sojourn (in the instant  $t_n$ ) of the wandering particle at the point  $x_j$ . Heuristically we consider  $y_j(t_n)$  as an approximation of  $h^N u(t_n, x_j) \approx \int_{C_j} u(t_n, x) dx$ , the total probability of sojourn inside a cubical cell  $C_j$  with the center  $x_j$  and side length  $h$ .

LEMMA 1. *For the probabilities  $y_j(t_n)$  the following statements hold true:*

- (i)  $y_j(t_{n+1}) = \sum_{k \in \mathbf{Z}^N} p_k y_{j-k}(t_n), j \in \mathbf{Z}^N;$
- (ii)  $y_j(t_n) = \underbrace{(p * \dots * p)}_{n \text{ times}}_j.$

In our further considerations we will need some properties of cubature formulas. Let  $f$  be a continuous function integrable over  $\mathbf{R}^N$ . Then the following rectangular cubature formula

$$\int_{\mathbf{R}^N} f(x) dx = h^N \sum_{j \in \mathbf{Z}^N} f(x_j) + o(1) \quad (1)$$

is valid [20].

### 3. Pseudo-differential operators and symbols

For our fractional diffusion processes we make essential use of the theory of pseudo-differential operators. For general orientation we recommend [5] and [9], furthermore the modern presentation in [10] by N. Jacob who pays special attention to Markov processes.

In this section we consider some properties of pseudo-differential operators  $A(D)$ ,  $D = (D_1, \dots, D_N)$ ,  $D_j = \frac{\partial}{i\partial x_j}$ ,  $j = 1, \dots, N$ , with a symbol  $A(\xi)$  not depending on  $x$  defined in  $\mathbf{R}^N$ . For a test function  $\varphi(x)$  taken from the classical space  $S(\mathbf{R}^N)$ , the Fourier transform

$$\hat{\varphi}(\xi) = F[\varphi](\xi) = \int_{\mathbf{R}^N} \varphi(x) e^{-ix\xi} dx$$

is well defined and belongs again to  $S(\mathbf{R}^N)$ . Let  $S'(\mathbf{R}^N)$  be the space of tempered distributions, i.e. the dual space to  $S(\mathbf{R}^N)$ . The Fourier transform for distributions  $f \in S'(\mathbf{R}^N)$  is usually defined by the extension formula  $(\hat{f}(\xi), \varphi(\xi)) = (f(x), \hat{\varphi}(x))$ , with the duality pair  $(\cdot, \cdot)$  of  $S'(\mathbf{R}^N)$  and  $S(\mathbf{R}^N)$ .

Assume  $G$  to be an open domain in  $\mathbf{R}^N$ . Let a function  $f$  be continuous and bounded on  $\mathbf{R}^N$  and have a Fourier transform (taken in the sense of distributions)  $\hat{f}(\xi)$  with compact support in  $G$ . The set of all such functions endowed with the convergence in the following sense is denoted by  $\Psi_G(\mathbf{R}^N)$ : a sequence of functions  $f_m \in \Psi_G(\mathbf{R}^N)$  is said to converge to an element  $f_0 \in \Psi_G(\mathbf{R}^N)$  iff: (i) there exists a compact set  $K \subset G$  such that  $\text{supp } \hat{f}_m \subset K$  for all  $m = 1, 2, \dots$ ; (ii)  $\|f_m - f_0\| = \sup |f_m - f_0| \rightarrow 0$  for  $m \rightarrow \infty$ . In the

case  $G = \mathbf{R}^N$  we write simply  $\Psi(\mathbf{R}^N)$  omitting  $\mathbf{R}^N$  in the index of  $\Psi_G(\mathbf{R}^N)$ . Note that according to the Paley-Wiener theorem functions in  $\Psi_G(\mathbf{R}^N)$  are entire functions of finite exponential type (see [15], [5]).

Let  $H^s(\mathbf{R}^N)$ ,  $s \in (-\infty, +\infty)$  be the Sobolev space of elements  $f \in S'(\mathbf{R}^N)$  for which  $(1 + |\xi|^2)^{s/2} |\hat{f}(\xi)| \in L_2(\mathbf{R}^N)$ . It is known [9] that if  $f \in L_p(\mathbf{R}^N)$  with  $p > 2$ , then its Fourier transform  $\hat{f}$  belongs to  $H^{-s}(\mathbf{R}^N)$ ,  $s > N(\frac{1}{2} - \frac{1}{p})$ . Letting  $p \rightarrow \infty$  we get  $\hat{f} \in H^{-s}(\mathbf{R}^N)$ ,  $s > \frac{N}{2}$  for  $f \in L_\infty(\mathbf{R}^N)$ . Taking into account this fact and the Paley-Wiener theorem we have that the Fourier transform of  $f \in \Psi_G(\mathbf{R}^N)$  belongs to the space

$$\bigcap_{s > \frac{N}{2}} H_{comp}^{-s}(G),$$

where  $H_{comp}^{-s}(G)$  is a negative order Sobolev space of functionals with compact support on  $G$ . Hence  $\hat{f}$  is a distribution, which is well defined on continuous functions.

Denote by  $\Psi'_{-G}(\mathbf{R}^N)$  the space of all linear bounded functionals defined on the space  $\Psi_G(\mathbf{R}^N)$  endowed with the weak (dual with respect to  $\Psi_G(\mathbf{R}^N)$ ) topology. Namely, we say that a sequence of functionals  $g_m \in \Psi'_{-G}(\mathbf{R}^N)$  converges to an element  $g_0 \in \Psi'_{-G}(\mathbf{R}^N)$  in the weak sense if for all  $f \in \Psi_G(\mathbf{R}^N)$  the sequence of numbers  $\langle g_m, f \rangle$  converges to  $\langle g_0, f \rangle$  as  $m \rightarrow \infty$ . By  $\langle g, f \rangle$  we mean the value of  $g \in \Psi'_{-G}(\mathbf{R}^N)$  on an element  $f \in \Psi_G(\mathbf{R}^N)$ .

Let  $A(\xi)$  be a continuous function defined in  $G \subset \mathbf{R}^N$ . A pseudo-differential operator  $A(D)$  with the symbol  $A(\xi)$  is defined by the formula

$$A(D)\varphi(x) = \frac{1}{(2\pi)^N} (\hat{\varphi}, A(\xi)e^{-ix\xi}) \quad (2)$$

which is well defined on  $\Psi_G(\mathbf{R}^N)$ . If  $\hat{\varphi}$  is an integrable function with  $\text{supp } \hat{\varphi} \subset G$ , then (2) becomes the usual form of pseudo-differential operator

$$A(D)\varphi(x) = \frac{1}{(2\pi)^N} \int A(\xi) \hat{\varphi}(\xi) e^{-ix\xi} d\xi,$$

with the integral taken over  $G$ . Note that in general this may not have sense even for infinitely differentiable functions with finite support (see [8]).

We define the operator  $A(-D)$  acting in the space  $\Psi'_{-G}(\mathbf{R}^N)$  by the extension formula

$$\langle A(-D)f, \varphi \rangle = \langle f, A(D)\varphi \rangle, \quad f \in \Psi'_{-G}(\mathbf{R}^N), \quad \varphi \in \Psi_G(\mathbf{R}^N). \quad (3)$$

We recall (see [8]) that the pseudo-differential operators  $A(D)$  and  $A(-D)$  with a continuous symbol  $A(\xi)$  act as

$$A(D) : \Psi_G(\mathbf{R}^N) \rightarrow \Psi_G(\mathbf{R}^N), A(-D) : \Psi'_{-G}(\mathbf{R}^N) \rightarrow \Psi'_{-G}(\mathbf{R}^N)$$

and are continuous.

LEMMA 2. *Let  $A(\xi)$  be a function continuous on  $\mathbf{R}^N$ . Then for  $\xi \in \mathbf{R}^N$*

$$A(D)\{e^{-ix\xi}\} = A(\xi)e^{-ix\xi}.$$

P r o o f. For any fixed  $\xi \in \mathbf{R}^N$  the function  $e^{-ix\xi}$  is in  $\Psi(\mathbf{R}^N)$ . We have

$$A(D)\{e^{-ix\xi}\} = \frac{1}{(2\pi)^N} \int_{\mathbf{R}^N} A(\eta)e^{-ix\eta} d\mu_\xi(\eta),$$

where  $d\mu_\xi(\eta) = F_\eta[e^{-ix\xi}]d\eta = (2\pi)^N \delta(\eta - \xi)d\eta$ . Hence  $A(D)\{e^{-ix\xi}\} = A(\xi)e^{-ix\xi}$ . ■

COROLLARY 1.

- i)  $A(\xi) = (A(D)e^{-ix\xi})e^{ix\xi}$ ;
- ii)  $A(\xi) = (A(D)e^{-ix\xi})|_{x=0}$ ;
- iii)  $A(\xi) = \langle A(-D)\delta(x), e^{-ix\xi} \rangle$ , where  $\delta$  is the Dirac distribution.

REMARK 1. Since the function  $e^{-ix\xi}$  does not belong to  $S(R^N)$  and  $D(R^N)$ , the representations for the symbol obtained in Lemma 2 and Corollary 1 are not applicable in these spaces.

In our further random walk constructions  $G = \mathbf{R}^N$ , so that symbols of pseudo-differential operators are continuous functions in the whole space. The symbol of the Laplace operator  $A(D) = \Delta$  as easily seen is  $-|\xi|^2$ . The relations in corollary for symbols can be easily verified for this symbol.

The pseudo-differential operator  $A(D) = D_0^\alpha$  with the symbol  $-|\xi|^\alpha$  can be represented with the help of a hypersingular integral (see, e.g. [18])

$$D_0^\alpha f(x) = -\frac{1}{d(\alpha, l)} \int_{\mathbf{R}^N} \frac{\Delta_y^l f(x)}{|y|^{N+\alpha}} dy, \quad (4)$$

where  $0 < \alpha < l$ ,  $l$  is a positive integer,  $\Delta_y^l$  is the finite difference of the order  $l$  in the  $y$  direction, either centered or non-centered, and  $d(\alpha, l)$  is a constant defined in dependence on what type of difference, centered or non-centered, is taken (see for details [18]). Note that in this paper we consider only the centered case of the finite difference  $\Delta_y^l$  in the definition of  $D_0^\alpha$ .

Let  $l$  be a given positive integer. Denote by  $\tau_y$  a shift operator with spatial vector-step  $y$

$$(\tau_y f)(x) = f(x - y), x, y \in \mathbf{R}^N.$$

Using this operator we determine the symmetric difference operator of order  $l$

$$(\Delta_y^l f)(x) = (\tau_{-\frac{y}{2}} - \tau_{\frac{y}{2}})^l f(x) = \sum_{k=0}^l (-1)^k \binom{l}{k} f(x + (\frac{l}{2} - k)y).$$

Let a constant  $d(\alpha, l)$  be defined as (see [18])

$$d(\alpha, l) = \frac{\pi^{1+N/2} A_l(\alpha)}{2^\alpha \Gamma(1 + \frac{\alpha}{2}) \Gamma(\frac{N+\alpha}{2}) \sin(\alpha\pi/2)}, \quad (5)$$

with  $A_l(\alpha)$  determined by the formula

$$A_l(\alpha) = 2 \sum_{k=0}^{[l/2]} (-1)^{k-1} \binom{l}{k} (\frac{l}{2} - k)^\alpha. \quad (6)$$

Moreover  $d(\alpha, l) \neq 0$  for all  $\alpha > 0$  and for even  $l$ , but  $d(\alpha, l)$  is identically zero for odd orders  $l$ . We accept that the hypersingular operator  $D_0^\alpha$  in (4) is defined with the so introduced  $\Delta_y^l$  and  $d(\alpha, l)$ .

In the construction of random walk models leading to the diffusion equations governed by pseudo-differential operators in space the sign of the normalizing constant  $d(\alpha, l)$  is important. Let  $l = 2$ . Then after slightly rearranging we have

$$D_0^\alpha f(x) = b(\alpha) \int_{\mathbf{R}^N} \frac{f(x - y) - 2f(x) + f(x + y)}{|y|^{N+\alpha}} dy, \quad (7)$$

where

$$b(\alpha) = \frac{\alpha \Gamma(\frac{\alpha}{2}) \Gamma(\frac{N+\alpha}{2}) \sin \frac{\alpha\pi}{2}}{2^{2-\alpha} \pi^{1+N/2}}. \quad (8)$$

It is seen from (8) that the value  $\alpha = 2$  is singular.

We note also that  $D_0^\alpha$  can be considered as a fractional power of the Laplace operator, namely  $D_0^\alpha = -(-\Delta)^{\alpha/2}$ . From Lemma 2 it follows that

$$D_0^\alpha e^{ix\xi}|_{x=0} = b(\alpha) \int_{\mathbf{R}^N} \frac{\Delta_y^2 e^{ix\xi}}{|y|^{N+\alpha}} dy|_{x=0} = -|\xi|^\alpha, 0 < \alpha < 2.$$

Taking into account the cubature formula (1) for the integral in the right hand side of (4) we have (with  $f_j = f(hj)$  and  $|k|$  as Euclidean norm of  $k = (k_1, \dots, k_N) \in \mathbf{Z}^N$ )

$$\int_{\mathbf{R}^N} \frac{\Delta_y^2 f(x_j)}{|y|^{N+\alpha}} dy = h^\alpha \sum_{k \in \mathbf{Z}^N} \frac{\Delta_k^2 f_j}{|k|^{N+\alpha}} + o(1).$$

#### 4. Fractional differential equations and random walk

Consider the fractional order diffusion equation

$$\frac{\partial}{\partial t} u(t, x) = D_0^\alpha u(t, x), \quad t > 0, \quad x \in \mathbf{R}^N, \quad (9)$$

where  $D_0^\alpha, 0 < \alpha < 2$ , is the pseudo-differential operator, defined in the previous section, which has the symbol  $-|\xi|^\alpha$ . In accordance with  $\lim_{\alpha \rightarrow 2} |\xi|^\alpha = |\xi|^2$  we will accept  $D_0^2 = \Delta$ , where  $\Delta$  is the Laplace operator. In the general case we have formally  $D_0^\alpha = -(-\Delta)^{\alpha/2}$ . A weak solution, namely a distribution  $G_\alpha(t, x)$ , which satisfies (9) and the condition

$$G_\alpha(0, x) = \delta(x), \quad x \in \mathbf{R}^N, \quad (10)$$

with the Dirac function  $\delta(x)$ , in the sense of distributions, is called a fundamental solution of the Cauchy problem (9), (10).

It is clear that in the case  $\alpha = 2$  we have the classical heat conduction equation

$$\frac{\partial}{\partial t} u(t, x) = \Delta u(t, x), \quad t > 0, \quad x \in \mathbf{R}^N,$$

whose fundamental solution is the Gauss probability density evolving in time

$$G_2(t, x) = \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}}.$$

In the case  $\alpha = 1$  the corresponding fundamental solution is given by the Cauchy-Poisson probability density (see [16])

$$G_1(t, x) = \frac{\Gamma(\frac{n+1}{2})}{\pi^{(n+1)/2}} \frac{1}{(|x|^2 + t^2)^{(n+1)/2}}.$$



It is well known that for the Fourier transforms of the functions  $G_q(t, x)$ ,  $q = 1, 2$ , the relations [16]

$$\hat{G}_2(t, \xi) = e^{-t|\xi|^2} \text{ and } \hat{G}_1(t, \xi) = e^{-t|\xi|}$$

hold.

For other values of  $\alpha$ ,  $0 < \alpha < 2$ , applying the Fourier transform and its inverse the fundamental solution to the Cauchy problem (9), (10) can be represented in the form

$$G_\alpha(t, x) = \frac{1}{(2\pi)^N} \int_{\mathbf{R}^N} e^{-t|\xi|^\alpha} e^{ix\xi} d\xi. \quad (11)$$

The question we want to explore is the existence of a random walk which approximates the diffusion process governed by the equation (9). In the random walk terminology this means that the fundamental solution, considered as a probability density, is the diffusion limit of some discrete random walk.

Bearing this in mind let us describe the discrete random walk model approximating the fractional diffusion process governed by the equation (9) in terms of probability theory. Let  $\mathbf{X}$  be an  $N$ -dimensional random vector [14] which takes values in  $\mathbf{Z}^N$ . Let the random vectors  $\mathbf{X}_1, \mathbf{X}_2, \dots$  also be  $N$ -dimensional independent identically distributed random vectors, all having their probability distribution common with  $\mathbf{X}$ . We introduce a spatial grid  $\{x_j = jh, j \in \mathbf{Z}^N\}$ , with  $h > 0$  and temporal grid  $\{t_n = n\tau, n = 0, 1, 2, \dots\}$  with a step  $\tau > 0$ . Consider the sequence of random vectors

$$\mathbf{S}_n = h\mathbf{X}_1 + h\mathbf{X}_2 + \dots + h\mathbf{X}_n, n = 1, 2, \dots$$

taking  $\mathbf{S}_0 = \mathbf{0}$  for convenience. We interpret  $\mathbf{X}_1, \mathbf{X}_2, \dots$ , as the jumps of a particle sitting in  $x = x_0 = \mathbf{0}$  at the starting time  $t = t_0 = 0$  and making a jump  $\mathbf{X}_n$  from  $\mathbf{S}_{n-1}$  to  $\mathbf{S}_n$  at the time instance  $t = t_n$ . Then the position  $\mathbf{S}(t)$  of the particle at time  $t$  is

$$\sum_{1 \leq k \leq t/\tau} \mathbf{X}_k.$$

Recall that the probability of sojourn of the particle in  $x_j$  at the time  $t_n$  was denoted by  $y_j(t_n)$ . Taking into account the recursion  $\mathbf{S}_{n+1} = \mathbf{S}_n + h\mathbf{X}_n$  we have

$$y_j(t_{n+1}) = \sum_{k \in \mathbf{Z}^N} p_k y_{j-k}(t_n), j \in \mathbf{Z}^N, n = 0, 1, \dots$$

The convergence of the sequence  $\mathbf{S}_n$  when  $n \rightarrow \infty$  means convergence of the discrete probability law  $(y_j(t_n))_{j \in \mathbf{Z}^N}$  properly rescaled as explained in the next Section, to the probability law with a density  $u(t, x)$  in the sense of distributions (in law). This is equivalent to the locally uniform convergence of the corresponding characteristic functions (see for details [14]). This will be used in the next section to prove the convergence of the constructed random walks to the fundamental solution of the governing diffusion equation (limit process).

### 5. Main result

Consider the Cauchy problem

$$\frac{\partial}{\partial t} u(t, x) = D_0^\alpha u(t, x), \quad t > 0, \quad x \in \mathbf{R}^N, \quad (12)$$

with the initial condition

$$u(0, x) = \delta(x), \quad x \in \mathbf{R}^N, \quad (13)$$

where  $0 < \alpha < 2$ ,  $D_0^\alpha$  is the pseudo-differential operator defined above with the symbol  $-|\xi|^\alpha$ ;  $D_0^2 = \Delta$ .

Assume that the hypersingular integral in the right hand side of (12) is defined by the centered finite difference of second order ( $l = 2$ ), i.e by the formula (7). Using the cubature formula for discrete approximation of the right hand side of (12), namely

$$D_0^\alpha u(t, x_j) \approx b(\alpha) \sum_{k \in \mathbf{Z}^N} \frac{u_{j+k}(t) - 2u_j(t) + u_{j-k}(t)}{|k|^{N+\alpha} h^\alpha},$$

where  $b(\alpha)$  is norming constant defined in (8), and replacing  $\frac{\partial u}{\partial t}$  by the first order difference ratio

$$\frac{\partial u}{\partial t} \approx \frac{u_j(t_{n+1}) - u_j(t_n)}{\tau}$$

with the time step  $\tau = t_{n+1} - t_n$  we have the relation

$$y_j(t_{n+1}) = \sum_{k \in \mathbf{Z}^N} p_k y_{j-k}(t_n)$$

which shows that the transition probabilities have the form

$$p_k = \begin{cases} 1 - \mu b(\alpha) \sum_{m \in \mathbf{Z}^N \setminus \{0\}} |m|^{-N-\alpha}, & \text{if } k = 0; \\ \mu b(\alpha) |k|^{-N-\alpha}, & \text{if } k \neq 0, \end{cases}$$

where the scaling parameter is  $\mu = \frac{2\tau}{h^\alpha}$ .

We will require that the transition probabilities satisfy the properties:

(i)  $\sum_{k \in \mathbf{Z}^N} p_k = 1$ ; (ii)  $p_k \geq 0, k \in \mathbf{Z}^N$ .

Now we formulate the main result of this paper. In the formulation we will use the notations introduced in the previous section.

**THEOREM .** *Let the transition probabilities  $p_k = P(\mathbf{X} = x_k), k \in \mathbf{Z}^N$ , of the random vector  $\mathbf{X}$  be given as follows:*

a) if  $0 < \alpha < 2$ , then

$$p_k = \begin{cases} 1 - \mu b(\alpha) \sum_{m \in \mathbf{Z}^N \setminus \{0\}} \frac{1}{|m|^{N+\alpha}}, & \text{if } k = 0; \\ \mu b(\alpha) |k|^{-(N+\alpha)}, & \text{if } k \neq 0, \end{cases}$$

with  $\mu$  satisfying the condition

$$0 < \mu \leq \frac{1}{b(\alpha) \sum_{m \in \mathbf{Z}^N \setminus \{0\}} |m|^{-N-\alpha}}.$$

and the space and time steps  $h$  and  $\tau$  being connected by the scaling relation  $\tau = \tau(h) = \mu h^\alpha / 2$ ;

b) if  $\alpha = 2$ , then

$$p_k = \begin{cases} \frac{1}{2N}, & \text{if } |k| = 1; \\ 0, & \text{if } |k| = 0, \end{cases}$$

with  $\tau = \frac{h^2}{2N}$ .

Then the sequence of random vectors  $\mathbf{S}_n = h\mathbf{X}_1 + \dots + h\mathbf{X}_n$ , converges as  $n \rightarrow \infty$  in the sense of distributions to the random vector whose probability density is the fundamental solution of the Cauchy problem (12), (13), i.e.  $G(t, x)$  defined in (11).

**P r o o f.** We have to show that the sequence of random vectors  $\mathbf{S}_n$  tends to the random vector with pdf

$$G(t, x) = \frac{1}{(2\pi)^N} \int_{\mathbf{R}^N} e^{-t|\xi|^\alpha} e^{ix\xi} d\xi.$$

It is obvious that the Fourier transform of  $G(t, x)$  with respect to the variable  $x$  is the function  $\hat{G}(t, \xi) = e^{-t|\xi|^\alpha}$ . Let  $\hat{p}(-\xi)$  be the characteristic function corresponding to the discrete function  $p_k$ ,  $k \in \mathbf{Z}^N$ , that is

$$\hat{p}(-\xi) = \sum_{k \in \mathbf{Z}^N} p_k e^{ik\xi}.$$

As a consequence of Lemma 1 and the well known fact that convolution goes over in multiplication by the Fourier transform, the characteristic function of  $y_j(t_n)$  can be represented in the form

$$\hat{y}_j(t_n, -\xi) = \hat{p}^n(-\xi).$$

Taking this into account it suffices to show that

$$\hat{p}^n(-h\xi) \rightarrow e^{-t|\xi|^\alpha}, n \rightarrow \infty. \quad (14)$$

The latter is equivalent to

$$\lim_{h \rightarrow 0} \frac{\ln \hat{p}(-h\xi)}{\tau(h)} = -|\xi|^\alpha.$$

where  $\tau(h) = \frac{t}{n} = \mu h^\alpha / 2$ . But for us it is more convenient to use the process (14). From the continuity of  $e^s$  it is readily seen that if a sequence  $s_n$  converges to  $s$  for  $n \rightarrow \infty$ , then

$$\lim (1 + \frac{s_n}{n})^n = e^s. \quad (15)$$

We have

$$\begin{aligned} \hat{p}^n(-h\xi) &= (1 - \frac{\mu b(\alpha)}{2} \sum_{k \in \mathbf{Z}^N \setminus \{0\}} \frac{2}{|k|^{N+\alpha}} (1 - e^{ik\xi h}))^n = \\ &= (1 + \frac{tb(\alpha) \sum_{k \in \mathbf{Z}^N \setminus \{0\}} \frac{\Delta^2 e^{ik\xi h}}{|kh|^{N+\alpha}} h^N}{n})^n. \end{aligned}$$

It follows from (1) and Lemma 2 (ii) that

$$b(\alpha) \sum_{k \in \mathbb{Z}^N} \frac{\Delta^2 e^{ik\xi h}}{|kh|^{N+\alpha}} h^N$$

tends to  $(D_0^\alpha e^{ix\xi})|_{x=0} = -|\xi|^\alpha$  as  $h \rightarrow 0$  (or, the same,  $n \rightarrow \infty$ ) for all  $\alpha \in (0, 2)$ . Hence in accordance with (15) we have

$$\hat{u}(t_n, -\xi) = \hat{p}^n(-h\xi) \rightarrow e^{-t|\xi|^\alpha}, n \rightarrow \infty.$$

The case  $\alpha = 2$  with the transition probabilities given in b) of Theorem can be treated as an exercise. ■

REMARK 2. The constructed random walk relates to the class of stable laws characteristic function of which in one-dimensional case is  $\exp(\psi(\xi))$  with

$$\psi(\xi) = ia\xi - b|\xi|^\alpha \{1 - i\beta \frac{\xi}{|\xi|} \omega(\xi, \alpha)\},$$

where  $a, b, \alpha, \beta$  are constants,  $a$  is real,  $b > 0$ ,  $0 < \alpha \leq 2$ ,  $-1 < \beta < 1$  and

$$\omega(\xi, \alpha) = \begin{cases} \tan(\frac{\pi}{2}\alpha), & \text{if } \alpha \neq 1; \\ \frac{2}{\pi} \log|\xi|, & \text{if } \alpha = 1. \end{cases}$$

The symmetric case corresponds to  $a = 0, \beta = 0$ . As is shown in [3] the governing equation of stable law is

$$\frac{\partial u}{\partial t} = -a \frac{\partial u}{\partial x} + Dq \frac{\partial^\alpha u}{\partial (-x)^\alpha} + Dp \frac{\partial^\alpha u}{\partial (x)^\alpha},$$

where  $D$  is some constant depending on  $\alpha$ ,  $p \geq 0$ ,  $q \geq 0$  and  $p + q = 1$ . The multidimensional analog of this equation as proposed in [13] is

$$\frac{\partial u(t, x)}{\partial t} = -a \nabla u(t, x) + D \nabla_M^\alpha u(t, x), x \in \mathbf{R}^N, t > 0,$$

where  $\nabla^\alpha$  is the pseudo-differential operator with the symbol

$$\int_{|\theta|=1} (-i\xi, \theta)^\alpha M(d\theta)$$

with  $M(d\theta)$ , a probability measure on the unit sphere. If  $a = 0$  and  $M(d\theta) = \text{const} \cdot d\theta$ , then we get the symmetric case considered above. The method demonstrated above for the symmetric case can be easily applied in the general case as well.

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